

RANDOM “DYADIC” LATTICE IN GEOMETRICALLY DOUBLING METRIC SPACE AND A_2 CONJECTURE

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ABSTRACT. Recently three proofs of the A_2 -conjecture were obtained. All of them are “glued” to euclidian space and a special choice of one random dyadic lattice. We build a random “dyadic” lattice in any doubling metric space which have properties that are enough to prove the A_2 -conjecture in these spaces.

1. INTRODUCTION

Our goal is to build an analog of the probability space of dyadic lattices, which was first constructed in [7, p. 207]. The main property of this space was that probability of a fixed cube to be “good” in a very precise sense is bounded away from zero.

Later in proofs, related to weighted Calderon-Zygmund theory, it was enough to consider a probability space whose elementary event was a pair of dyadic lattices. This ideology was used in [8] and [7]. In the case of a geometrically doubling metric space (GDMS) Michael Christ, [1] introduced dyadic “lattice”, and later Hytönen and Martikainen, [3], randomized Christ’s construction. They also have a probability space, which consists of pairs of dyadic lattices on the GDMS. The definition of a “good” cube in one lattice was given using the second lattice.

However, recently the solution of so called A_2 conjecture in Euclidian setting essentially required the construction from [7, p. 207], where the elementary event is **one** dyadic lattice, and not a pair of them. The definition of a “good” cube in this lattice was given in terms of other cubes of the **same** lattice. All the proofs of A_2 conjecture so far were based on decomposition of a Calderon-Zygmund operator to dyadic shifts and thus required this **one** lattice randomization.

The first proof of A_2 conjecture was given by Hytönen, in which he essentially used reduction to the weighted $T1$ theorem from [11]. The new tool — in comparison to [7] — is that one can, instead of estimating the contribution of “bad” cubes (as in [7]), just ignore this contribution completely. For that one needed (we repeat) the construction of probability space, consisting of **one** dyadic lattice as an elementary event. Also one needs that the probability to be “good” is the same for all cubes.

Immediately after this proof a simplified proof was given in [4]. It again uses the same properties of probability space of dyadic lattices, but the decomposition of the operator to dyadic shifts was simplified, and instead of the weighted $T1$ theorem for an arbitrary Calderon-Zygmund operator ([11]) one used $T1$ theorem for shifts, [9].

Another Bellman function proof of the same A_2 conjecture was recently obtained in [10].

In the present work we build a probability space of dyadic “lattices” in a GDMS, where an elementary event is one lattice, and the probability to be “good” is the same for all cubes in the lattice. As an application, with a combination of this idea

and tools from any of [2], [4], or [10] (the last one will give the shortest proof), we obtain a proof of the A_2 conjecture in an arbitrary geometrically doubling metric space.

2. ACKNOWLEDGEMENTS

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3. FIRST STEP

Consider a doubling metric space X with metric d and doubling constant A . Instead of $d(x, y)$ we write $|xy|$.

As authors of [3], we essentially use the idea of Michael Christ, [1], but randomize his construction in a different way. Therefore, we want to guard the reader that even though on the surface the proof below is very close to the proof from [3], however, our construction is essentially different, and so the proof of the assertion in our main lemma, which was not hard in [3], becomes much more subtle here.

For a number $k > 0$ we say that a set G is a k -grid if G is maximal set, such that for any $x, y \in G$ we have $d(x, y) \geq k$.

Take a small positive number δ and a large natural number N , and for every $M \geq N$ fix $G_M = \{z_M^\alpha\}$ — a δ^M -grid. Now take G_N and randomly choose a $G_{N-1} = \delta^{N-1}$ -grid in G_N . Then take G_{N-1} and randomly choose a $G_{N-2} = \delta^{N-2}$ -grid there.

Lemma 3.1.

$$\bigcup_{y \in G_{N-k}} B(y, 3\delta^{N-k}) = X.$$

(for $N + k$ this is obvious)

Proof. Take $x \in X$

Then, since G_N is maximal, there exists a point $y_0 \in G_N$, such that $|xy_0| \leq \delta^N$. Since G_{N-1} is maximal in G_N , there is a point $y_1 \in G_{N-1}$, such that $|y_0y_1| \leq \delta^{N-1}$. Similarly we get y_2, \dots, y_k and then

$$|xy_k| \leq |xy_0| + \dots + |xy_k| \leq \delta^N + \dots + \delta^{N-k} = \delta^{N-k}(1 + \delta + \dots + \delta^k) \leq \frac{\delta^{N-k}}{1 - \delta} \leq 2\delta^{N-k}.$$

□

Once we have all our sets G_N , we introduce a relationship \prec between points. We follow [3] and [1].

Take a point $y_{k+1} \in G_{k+1}$. There exists at most one $y_k \in G_k$, such that $|y_{k+1}y_k| \leq \frac{\delta^k}{4}$. This is true since if there are two such points y_k^1, y_k^2 , then

$$|y_k^1 y_k^2| \leq \frac{\delta^k}{2},$$

which is a contradiction, since G_k was a δ^k -grid in G_{k+1} .

Also there exists at least one $z_k \in G_k$ such that $|y_{k+1}z_k| \leq 3\delta^k$. This is true by the lemma.

Now, if there exists an y_k as above, we set $y_{k+1} \prec y_k$. If no, then we pick one of z_k as above and set $y_{k+1} \prec z_k$. For all other $x \in G_k$ we set $y_{k+1} \not\prec x$. Then extend by transitivity.

We also assume that $y_k \prec y_k$.

We do this procedure randomly and independently, and treat same families of G_k 's with different \prec -law as different families.

Take now a point $y_k \in G_k$ and define

$$Q_{y_k} = \bigcup_{z \prec y_k, z \in G_\ell} B(z, \frac{\delta^\ell}{100}).$$

Lemma 3.2. *For every k we have*

$$X = \bigcup_{y_k \in G_k} \text{clos}(Q_{y_k})$$

Proof. Take any $x \in X$. By the previous lemma, for every $m \geq k$ there exists a point $x_m \in G_m$, such that $|xx_m| \leq 3\delta^m$. In particular, $x_m \rightarrow x$. Fix for a moment x_m . Then there are points $y_{m-1} \in G_{m-1}, \dots, y_k \in G_k$, such that $x_m \prec y_{m-1} \prec \dots \prec y_k$. In particular, $x_m \in Q_{y_k}$, where y_k depends on x_m . Then

$$|y_k x| \leq |y_k x_m| + |x_m x| \leq |y_k x_m| + 3\delta^m \leq |y_k x_m| + 3\delta^k.$$

Moreover, by the chain of \prec 's, we know that $|y_k x_m| \leq 10\delta^k$. Therefore,

$$|y_k x| \leq 15\delta^k.$$

We claim that the set $\{y_k\} = \{y_k(x_m)\}_{m \geq k}$ is finite. This is true since all y_k 's are separated from each other and by the doubling of our space (we are “stuffing” the ball $B(x, 15\delta^k)$ with balls $B(y_k, \delta^k)$).

So, take an infinite subsequence x_m that corresponds to one point $y_k \in G_k$. Then we get $x_m \in Q_{y_k}$, $x_m \rightarrow x$, so $x \in \text{clos} Q_{y_k}$, and we are done. \square

4. SECOND STEP: TECHNICAL LEMMATA

Define

$$\tilde{Q}_{y_k} = X \setminus \bigcup_{z_k \neq y_k, z_k \in G_k} \text{clos} Q_{z_k}.$$

In particular,

$$Q_{y_k} \subset \tilde{Q}_{y_k} \subset \text{clos}(Q_{y_k}).$$

Lemma 4.1 (Lemma 4.5 in [3]). *Let m be a natural number, $\varepsilon > 0$, and $\delta^m \geq 100\varepsilon$. Suppose $x \in \text{clos} Q_{y_k}$ and $\text{dist}(x, X \setminus \tilde{Q}_{y_k}) < \varepsilon\delta^k$. Then for any chain*

$$z_{k+m} \prec z_{k+m-1} \prec \dots \prec z_{k+1} \prec z_k,$$

such that $x \in \text{clos} Q_{z_{k+m}}$, there holds

$$|z_i z_j| \geq \frac{\delta^j}{100}, \quad k \leq j < i \leq k+m.$$

Proof. Suppose $|z_i z_j| < \frac{\delta^j}{100}$. We first consider a case when $z_k = y_k$. Since $z_j \prec z_k = y_k$, we have $B(z_j, \frac{\delta^j}{200}) \subset Q_{y_k} \subset \tilde{Q}_{y_k}$. Therefore,

$$\frac{\delta^j}{200} \leq \text{dist}(z_j, X \setminus \tilde{Q}_{y_k}) \leq \text{dist}(x, X \setminus \tilde{Q}_{y_k}) + \text{dist}(x, z_i) + \text{dist}(z_i, z_j) < \varepsilon\delta^k + 5\delta^i + \frac{\delta^j}{100}$$

If δ is less than, say, $\frac{1}{1000}$, then we get a contradiction.

The only not obvious estimate is that $\text{dist}(x, z_i) < 5\delta_i$. It is true since $x \in \text{clos}Q_{z_{k+m}}$.

We have proved the lemma with assumption that $z_k = y_k$. Let us get rid of this assumption. We know that

$$x \in \text{clos}Q_{z_{k+m}} \subset \text{clos}Q_{z_k}.$$

Also we have $x \in \text{clos}Q_{y_k}$, so, since

$$\tilde{Q}_{z_k} = X \setminus \bigcup_{u_k \neq z_k} \text{clos}Q_{u_k} \subset X \setminus \text{clos}Q_{y_k},$$

we get $x \in X \setminus \tilde{Q}_{z_k}$. In particular, $\text{dist}(x, X \setminus \tilde{Q}_{z_k}) = 0 < \varepsilon\delta^k$, and we are in the situation of the first part. This finishes our proof. \square

Lemma 4.2 (MAIN LEMMA). *Fix $x_k \in G_k$. Then*

$$(1) \quad \mathbb{P}(\exists x_{k-1} \in G_{k-1} : |x_k x_{k-1}| < \frac{\delta^{k-1}}{1000}) \geq a$$

for some $a \in (0, 1)$.

Proof. To illustrate the proof we consider a particular case, but with a stronger result.

Consider X to be a tree with a distance between to neighbor vertices equal to one. The geometrically doubling condition can be expressed in the following way: each vertex has no more that C sons, where C is a fixed number. For the sake of simplicity we consider $C = 3$.

We now randomly choose a 2-grid G . In other words, we color our tree in two colors, red and green, such that

- If a vertex is red then all its neighbors are green;
- The set of red vertices is maximal, i.e., if a vertex is green there is a red vertex on distance 1.

We proof a stronger result: fix a vertex z then

$$\mathbb{P}(\exists x \in G : x = z) > a.$$

Without loss of generality, we consider that z is a root and it has 3 sons. Consider z and assume that it is red. We introduce a procedure of recoloring our tree such that z becomes green. If z is red then all its sons, z_1, z_2, z_3 , are green. If one of them has only green sons then everything is easy: we color z in green color and this son in red.

Suppose all $z_{1,2,3}$ have one red son. Then we proceed by induction (the base, when height of our tree is 2, is trivial). Take z_1 and its sons z_{11}, z_{12}, z_{13} . If z_{11} is red then we recolor all the sub-tree of z_{11} , such that z_{11} becomes green. We do the same thing with z_{12} and z_{13} . After that we can color z_1 in red and z in green.

Notice that this procedure has the following properties: for two different initial coloring in gives different coloring. So we proved that

$$\mathbb{P}(z \text{ is green}) \geq \frac{1}{2}.$$

Now,

$$\mathbb{P}(z \text{ is red}) = \mathbb{P}(z_{1,2,3} \text{ are green}),$$

so

(2)

$$\frac{\#\{\text{colorings with red } z\}}{\#\{\text{all colorings}\}} = \frac{\#\{\text{colorings with green } z_1\}}{\#\{\text{all colorings of } z_1\text{-subtree}\}} \frac{\#\{\text{colorings with green } z_2\}}{\#\{\text{all colorings of } z_2\text{-subtree}\}} \cdot \frac{\#\{\text{colorings with green } z_3\}}{\#\{\text{all colorings of } z_3\text{-subtree}\}}.$$

$$\frac{\#\{\text{all colorings of } z_1\text{-subtree}\} \cdot \#\{\text{all colorings of } z_2\text{-subtree}\} \cdot \#\{\text{all colorings of } z_3\text{-subtree}\}}{\#\{\text{all colorings}\}}$$

First three fractions are bigger than $\frac{1}{2}$ by the recoloring argument. The last one is not equal to one (getting a coloring the whole tree, we may not get a proper coloring of, say z_1 -subtree), but is also bigger than $\frac{1}{2}$. Here is the reason:

$$\#\{\text{all colorings}\} \leq 2\#\{\text{colorings with green } z\},$$

and colorings with green z always give a proper coloring of $z_{1,2,3}$ -subtrees.

So our probability is bigger than $\frac{1}{16}$, which finishes the proof. \square

Remark 1. *All numbers from this proof, such as distances 1 and 2, number of sons, can be changed to arbitrary fixed numbers.*

After this illustration let us give the proof of the main lemma in full generality.

Proof. So let us be in a compact metric situation. By rescaling we can think that we work with G_1 and choose G_0 . We can even think that the metric space consists of finitely many points, it is $X := G_2$. The finite set $G_1 \subset X$ consists of points having the following properties:

1. $\forall x, y \in G_1$ we have $|xy| \geq \delta$;
2. if $z \in X \setminus G_1$ then $\exists x \in G_1$ such that $|zx| < \delta$.

These two properties are equivalent to saying that the subset G_1 of X consists of points such that $\forall x, y \in G_1$ we have $|xy| \geq \delta$ and we cannot add any point from X to G_1 without violating that property. In other words: G_1 is a *maximal* set with property 1.

Here the word “maximal” means maximal with respect to inclusion, not maximal in the sense of the number of elements.

Now we consider the new metric space $Y = G_1$ and G_0 is any maximal subset such that

$$(3) \quad \forall x, y \in G_0, |xy| \geq 1.$$

In other words, we have 1. $\forall x, y \in G_0$ we have $|xy| \geq 1$;

2. if $z \in Y \setminus G_0$ then $\exists x \in G_0$ such that $|zx| < 1$.

There are finitely many such maximal subsets G_0 of Y . We prescribe for each choice the same probability. Now we want to prove the claim that is even stronger than (1). Namely, we are going to prove that given $y \in Y$

$$(4) \quad \mathbb{P}(\exists x_0 \in G_0: x_0 = y) \geq a,$$

where a depends only on δ and the constants of geometric doubling of our compact metric space.

Let Y be any metric space with finitely many elements. We will color the points of Y into red and green colors. The coloring is called proper if

1. every red point does not have any other red point at distance < 1 ;
2. every green point has at least one red point at distance < 1 .

Given a proper coloring of Y the collection of red points is called 1-lattice. It is a maximal (by inclusion) collection of points at distance ≥ 1 from each other.

What we need to finish the main lemma's proof is

Lemma 4.3. *Let Y be a finite metric space as above. Assume Y has the following property:*

- (5) *In every ball of radius less than 1 there are at most d elements.*

Let \mathcal{L} be a collection of 1-lattices in Y . Elements of \mathcal{L} are called L . Let $v \in Y$. Then

$$\frac{\text{the number of 1-lattices } L \text{ such that } v \text{ belongs to } L}{\text{the total number of 1-lattices } L} \geq a > 0,$$

where a depends only on d .

Proof. Given $v \in Y$ consider all subsets of $B(v, 1) \setminus v$, this collection is called \mathcal{S} . Let $S \in \mathcal{S}$. We call W_S the collection of all proper colorings such that v is green, all elements of S are red, and all elements of $B(v, 1) \setminus S$ are green. We call \tilde{S} all points in Y , which are not in $B(v, 1)$, but at distance < 1 from some point in S .

All proper colorings of Y such that v is red are called B . Let us show that

- (6) $\text{card } W_S \leq \text{card } B$.

Notice that if (6) were proved, we would be done with Lemma 4.3, $a \geq 2^{-d+1}$, and, consequently, the proof of the main lemma would be finished, $a \geq 2^{-\delta-D}$, where D is a geometric doubling constant.

To prove (6) let us show that we can recolor any proper coloring from W_S into the one from B , and that this map is injective. Let $L \in W_S$. We

1. Color v into red;
2. Color S into green;
3. Elements of \tilde{S} were all green before. We leave them green, but we find among them all those y that now in the open ball $B(y, 1)$ in Y all elements are green. We call them yellow (temporarily) and denote them Z ;
4. We enumerate Z in any way (non-uniqueness is here, but we do not care);
5. In the order of enumeration color yellow points to red, ensuring that we skip recoloring of a point in Z if it is at < 1 distance to any previously colored yellow-to-red point from Z . After several steps all green and yellow elements of \tilde{S} will have the property that at distance < 1 there is a red point;
6. Color the rest of yellow (if any) into green and stop.

We result in a proper coloring (it is easy to check), which is obviously B . Suppose L_1, L_2 are two different proper coloring in W_S . Notice that the colors of $v, S, B(v, 1) \setminus S, \tilde{S}$ are the same for them. So they differ somewhere else. But our procedure does not touch "somewhere else". So the modified colorings L'_1, L'_2 that we obtain after the algorithm 1-6 will differ as well may be even more. So our map $W_S \rightarrow B$ (being not uniquely defined) is however injective. We proved (6).

□

□

5. MAIN DEFINITION AND THEOREM

Definition 1 (Bad cubes). Take a “cube” $Q = Q_{x_k}$. We say that Q is good if there exists a cube $Q_1 = Q_{x_n}$, such that if

$$\delta^k \leq \delta^r \delta^n \quad (k \geq n + r)$$

then either

$$\text{dist}(Q, Q_1) \geq \delta^{k\gamma} \delta^{n(1-\gamma)}$$

or

$$\text{dist}(Q, X \setminus Q_1) \geq \delta^{k\gamma} \delta^{n(1-\gamma)}.$$

Remark 2. Notice that $\delta^k = \ell(Q)$ just by definition.

If Q is not good we call it bad.

Theorem 5.1. Fix a cube Q_{x_k} . Then

$$\mathbb{P}(Q_{x_k} \text{ is bad}) \leq \frac{1}{2}.$$

Remark 3 (Discussion). This theorem makes sense because when we fix a cube Q_k , say, $k \geq N$, so the grid G_k is not even random, we can make big cubes random. And we claim that for big quantity of choices, our big cubes will have Q_k either “in the middle” or far away, but not close to the boundary.

Definition 2. For $Q = Q_k$ define

$$\delta_Q(\varepsilon) = \delta_Q = \{x : \text{dist}(x, Q) \leq \varepsilon \delta^k \text{ and } \text{dist}(x, X \setminus Q) \leq \varepsilon \delta^k\}$$

Lemma 5.2.

$$\mathbb{P}(x \in \delta_{Q_k}) \leq \varepsilon^\eta$$

for some $\eta > 0$.

Proof of the theorem. Take the cube Q_{x_k} . There is a unique (random!) point x_{k-s} such that $x_k \in Q_{x_{k-s}}$. Then

$$\text{dist}(Q_{x_k}, X \setminus Q_{x_{k-s}}) \geq \text{dist}(x_k, X \setminus Q_{x_{k-s}}) - \text{diam}(Q_{x_k}) \geq \text{dist}(x_k, X \setminus Q_{x_{k-s}}) - C\delta^k.$$

Assume that $\text{dist}(x_k, X \setminus Q_{x_{k-s}}) > 2\delta^{k\gamma} \delta^{(k-s)(1-\gamma)}$ and that $s \geq r$ (this assumption is obvious, otherwise $Q_{x_{k-s}}$ does not affect goodness of Q_{x_k}).

Then, if r is big enough ($\delta r(1-\gamma) < \frac{1}{2}$) we get

$$\text{dist}(Q_{x_k}, X \setminus Q_{x_{k-s}}) \geq \delta^{k\gamma} \delta^{(k-s)(1-\gamma)},$$

and so Q_{x_k} is good. Therefore,

$$\mathbb{P}(Q_{x_k} \text{ is bad}) \leq C \sum_{s \geq r} \mathbb{P}(x_k \in \delta_{Q_{x_{k-s}}}(\varepsilon = \delta^{s\gamma})) \leq C \sum_{s \geq r} \delta^{\eta\gamma s} \leq 100C\delta^{\eta\gamma r}.$$

For sufficiently large r (or small δ) this is less than $\frac{1}{2}$. \square

Remark 4 (Discussion). At the end of the proof we have claimed that

$$\mathbb{P}(Q_{x_k} \text{ is bad}) \leq C \sum_{s \geq r} \mathbb{P}(x_k \in \delta_{Q_{x_{k-s}}}(\varepsilon = \delta^{s\gamma})) \leq C \sum_{s \geq r} \delta^{\eta\gamma s} \leq 100C\delta^{\eta\gamma r}.$$

In particular, we did some estimate of the probability that $x_k \in \delta_{Q_{x_{k-s}}}$. Here it is crucial that the point x_k is fixed and not random, so in some sense cubes $Q_{x_{k-s}}$ does

not depend on x_k (the conditional probability is equal to the unconditional probability, since Q_k is fixed and not random).

Proof of the lemma. Fix the largest m such that $500\varepsilon \leq \delta^m$. Choose a point x_{k+m} such that $x \in Q_{x_{k+m}}$. Then by the main lemma

$$\mathbb{P}(\exists x_{k+m-1} \in G_{k+m-1} : |x_{k+m}x_{k+m-1}| < \frac{\delta^{k+m-1}}{1000}) \geq a.$$

Therefore,

$$\mathbb{P}(\forall x_{k+m-1} \in G_{k+m-1} : |x_{k+m}x_{k+m-1}| \geq \frac{\delta^{k+m-1}}{1000}) \leq 1 - a.$$

Let now

$$x_{k+m} \prec x_{k+m-1}.$$

Then

$$\mathbb{P}(\forall x_{k+m-2} \in G_{k+m-2} : |x_{k+m-1}x_{k+m-2}| \geq \frac{\delta^{k+m-2}}{1000}) \leq 1 - a.$$

So

$$\mathbb{P}(\text{dist}(x, X \setminus \tilde{Q}_k) < \varepsilon \delta^k) \leq \mathbb{P}(|x_{k+j}x_{k+j-1}| \geq \frac{\delta^{k+j-1}}{1000} \forall j = 1, \dots, m) \leq (1-a)^m \leq C\varepsilon^\eta$$

for

$$\eta = \frac{\log(1-a)}{\log(\delta)}.$$

□

6. PROBABILITY TO BE “GOOD” IS THE SAME FOR EVERY CUBE

We make the last step to make the probability to be “good” not just bounded away from zero, but the same for all cubes. We use the idea from [5].

Take a cube $Q(\omega)$. Take a random variable $\xi_Q(\omega')$, which is equally distributed on $[0, 1]$. We know that

$$\mathbb{P}(Q \text{ is good}) = p_Q > a > 0.$$

We call Q “really good” if

$$\xi_Q \in [0, \frac{a}{p_Q}].$$

Otherwise Q joins bad cubes. Then

$$\mathbb{P}(Q \text{ is really good}) = a,$$

and we are done.

7. APPLICATION

As a main application of our construction, we state the following theorem.

Definition 3. Let X be a geometrically doubling metric space. Suppose $K(x, y): X \times X \rightarrow \mathbb{R}$ is a Calderon-Zygmund kernel of singularity m , i. e.

$$(7) \quad |K(x, y)| \leq \frac{1}{|xy|^m}$$

$$(8) \quad |K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \frac{|xx'|^\varepsilon}{|xy|^{m+\varepsilon}}.$$

Let μ be a measure on X , such that $\mu(B(x, r)) \leq Cr^m$, where C doesn't depend on x and r . We say that T is a Calderon-Zygmund operator with kernel K if

$$(9) \quad T \text{ is bounded } L^2(\mu) \rightarrow L^2(\mu),$$

$$(10) \quad Tf(x) = \int K(x, y)f(y)d\mu(y), \quad \forall x \notin \text{supp}\mu.$$

Definition 4. Let $w > 0$ μ -a.e. Define

$$w \in A_{2,\mu} \Leftrightarrow [w]_{2,\mu} = \sup_B \frac{1}{\mu(B)} \int_B w d\mu \cdot \frac{1}{\mu(B)} \int_B w^{-1} d\mu < \infty.$$

Theorem 7.1 (A_2 theorem for a geometrically doubling metric space). Let X be a geometrically doubling metric space, μ and T as above, $w \in A_{2,\mu}$. In addition we assume that μ is a doubling measure. Then

$$\|T\|_{L^2(wd\mu) \rightarrow L^2(wd\mu)} \leq C(T)[w]_{2,\mu}.$$

Remark 5. We note that existence of such μ in a GDMS was proved in [6].

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